Note to Copula Functions[1]

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Abstract

The theory of copulas is known to provide a useful tool for modelling dependence in integrated risk management. In this paper, we describe how may be used copula methodology for the Monte Carlo Analysis whereas the main emphasis is put on Value-at-Risk as a risk measure. In the second part of this paper we show properties more generalised model as measurable space and we show how it is possible to introduce the relevant notions as for example the joint distribution.

Key words: copula, correlation, Value-at-Risk, Monte Carlo Analysis, orthomodular lattice, state, observable, s-map

1 Introduction

Capital allocation within a bank is getting more and more important as the regulatory requirements are moving towards economic-based measures of risk (see the reports [1] and [2]). Banks are urged to build sound internal measures of credit and market risks for all their activities (and certainly for operational risk in a near future). Internal models for credit, market and operational risks are fundamental for bank capital allocation in a *bottom-up* approach. Internal models generally face an important problem, which is the modelling of joint distributions of different risks.

These two difficulties (Gaussian and joint distribution modelling) can be treated as a problem of copulas. A copula is a function that links univariate marginals to their multivariate distribution. Before 1999, copulas have not been used in finance. There have been recently some interesting papers on this subject (see for example the article of Embrechts, McNeil and Straumann [1999]). Moreover, copulas are more often cited in the financial literature. Li [1999] studies the problem on default correlation in credit risk models, and shows that "the current CreditMetrics approach to default correlation through asset correlation is equivalent to using a normal copula function". In the Risk special report of November 1999 on Operational Risk, Ceske and Hernández[1999] explain that copulas may be used in conjunction with Monte Carlo methods to aggregate correlated losses.

The aim of this paper is to show how may be used copula methodology for the Monte Carlo Analysis whereas the main emphasis is put on Value-at-Risk as a risk measure. The paper is organized as follows. In section two, we present copula function and some related fields. In the section three we consider the problem of

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Monte Carlo simulating analysis and explain that copulas may be used in conjunction with Monte Carlo methods to aggregate correlated losses. In the last section four we introduce more general model for random events as Boolean algebra and we discus about the basic properties of such model.

2. Copulas approaches

The problem of risk measuring for a financial asset portfolio may be divided into two main stages:

- modelling the joint evolution of risk factor[2] returns affecting portfolio's profit and loss distribution over a specified holding period[3];
- modelling the impact of risk factor return changes on the value of assets[4] in portfolio by using adequate pricing models.

In this work, the focus is on the first stage of the problem. We describe (in a practical way) how to simulate the risk factors from a multivariate distribution. In order to achieve this purpose, it is necessary to know the dependence structure of risk factor returns. Let $\boldsymbol{X}=(X_1, ..., X_n)$ be the random vector of the n risk factor log-returns which affect portfolio value, with marginal cumulative distribution functions (C.D.F.) $F_1, ..., F_n$. The multivariate C.D.F., $\boldsymbol{F}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n) = \boldsymbol{P}[\boldsymbol{X}_1 \le \boldsymbol{x}_1,\ldots,\boldsymbol{X}_n \le \boldsymbol{x}_n]$, completely determines the dependence structure of random returns $X_1, ..., X_n$. However, its analytic representation is often too complex, making practically impossible its estimation and consequently its use in simulation models. The most common methodologies for measuring portfolio risk use the multivariate conditional Gaussian distribution to simulate risk factor returns due to its easy implementation. Unfortunately, empirical evidence underlines its inadequacy in fitting real data. The use of copula function allows us to overcome the issue of estimating the multivariate C.D.F. by splitting it into two parts:

- determine the margins $F_1, ..., F_n$, representing the distribution of each risk factor; estimate their parameters fitting the available data by soundness statistical methods[5];
- determine the dependence structure of the random variables $X_1, ..., X_n$, specifying a meaningful copula function.

The main goal is to choose the margins and the copula better performing the portfolio Value-at-Risk (VaR) measurement.

2.1 Some definitions and properties

Definition 1: An n-dimensional copula[6] is a multivariate C.D.F., $\boldsymbol{C}$, with uniformly distributed margins on [0,1] ($\boldsymbol{U}(0,1)$) and the following properties:

1. $\boldsymbol{C}$: $[0,1]^n \rightarrow [0,1]$;
2. $\boldsymbol{C}$ is grounded and n-increasing[7];
3. $\boldsymbol{C}$ has margins $\boldsymbol{C_i}$ which satisfy $\boldsymbol{C_i}(u) = \boldsymbol{C}(1, ..., 1, u, 1, ..., 1) = u$ for all $u \in [0,1]$.

It is obvious, from the above definition, that if $\boldsymbol{F_1, ..., F_n}$ are univariate distribution functions, $\boldsymbol{C}(\boldsymbol{F}_1(\boldsymbol{x}_1),\ldots,\boldsymbol{F}_n(\boldsymbol{x}_n))$ is a multivariate C.D.F. with margins $\boldsymbol{F_1, ..., F_n}$, since $\boldsymbol{U}_i = \boldsymbol{F}_i(\boldsymbol{X}_i)$, $i = 1,...,n$, is a uniform random variable. Copula functions are a useful tool to construct and simulate multivariate distributions.

The following theorem is known as **Sklar's Theorem**. It is the most important theorem regarding to copula functions since it is used in many practical applications.

Theorem[8]: *Let F be an n-dimensional C.D.F. with continuous margins $F_1, ..., F_n$. Then F has the following unique copula representation (canonical decomposition):*

$$\boldsymbol{F}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n) = \boldsymbol{C}(\boldsymbol{F}_1(\boldsymbol{x}_1),\ldots,\boldsymbol{F}_n(\boldsymbol{x}_n)) \qquad (2.1.1)$$

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[2] e.g. , exchange rates, interest rates, stock indexes, commodity prices, and risk factors affecting the credit state of the counterparty.

[3] Usually it ranges from one day to two weeks for market risk management, while it is one year for credit risk management.

[4] Such as options, swaps, bonds, equities, etc.

[5] e.g., Generalized Method of Moments (GMM), Maximum Likelihood Estimation (MLE)., etc.

[6] The original definition is given by Sklar (1959).

[7] These properties mean that $\boldsymbol{C}$ is a positive probability measure.

[8] For the proof, see Sklar (1996).

The theorem of Sklar [1959] is very important, because it provides a way to analyse the dependence structure of multivariate distributions without studying marginals distributions. From Sklar's theorem we see that, for continuous multivariate distribution functions, the univariate margins and the multivariate dependence

structure can be separated. The dependence structure can be represented by an adequate copula function. Moreover, the following corollary is attained from (2.1.1).

Corollary: *Let* $\boldsymbol{F}$ *be an n-dimensional C.D.F. with continuous margins* $\boldsymbol{F}_1, ..., \boldsymbol{F}_n$ *and copula* $\boldsymbol{C}$ *(satisfying* (2.1.1)*). Then, for any* $\boldsymbol{u}=(u_1,...,u_n)$ *in* $[0,1]^n$*:*

$$\boldsymbol{C}(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n) = \boldsymbol{F}(\boldsymbol{F}_1^{-1}(\boldsymbol{u}_1),\ldots,\boldsymbol{F}_n^{-1}(\boldsymbol{u}_n)) \qquad (2.1.2)$$

where $\boldsymbol{F}_i^{-1}$ is the generalized inverse of $\boldsymbol{F}_i$**.**

To illustrate the idea behind the copula function, we can think about the multivariate Gaussian that is a "standard" assumption in risk management.

Corollary: The Gaussian (or normal) copula is the copula of the multivariate normal distribution. In fact, the random vector $\boldsymbol{X}=(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n)$ is multivariate normal iff:

1) the univariate margins $\boldsymbol{F}_1, ..., \boldsymbol{F}_n$ are Gaussians;

2) the dependence structure among the margins is described by a unique copula function $\boldsymbol{C}$ (the normal copula) such that[9]:

$$\boldsymbol{C}_{\boldsymbol{R}}^{Ga}(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n) = \Phi_{\boldsymbol{R}}(\phi^{-1}(\boldsymbol{u}_1),\ldots,\phi^{-1}(\boldsymbol{u}_n)) \qquad (2.1.3)$$

where $\Phi_{\boldsymbol{R}}$ is the standard multivariate normal C.D.F. with linear correlation matrix **R** and φ^{-1} is the inverse of the standard univariate Gaussian C.D.F.

It appears that the risk can be splitted into two parts: the individual risks and the dependence structure between them. Indeed, the assumption of normality for the margins can be removed and $\boldsymbol{F}_1, ..., \boldsymbol{F}_n$ may be fat-tailed distributions (e.g. Student, Weibull, Pareto) and dependence may still be characterized by a Normal copula. This separability property is not (just!) a piece of obtuse statistical theory, but has profound and far-reaching practical implications. These include the following:

- Copulas provide greater flexibility in that they allow us to fit any marginals we like to different random variables, and these distributions might differ from one variable to another. We might fit a normal distribution to one variable and another distribution to the second, and then fit any copula we like across the marginals. In contrast, traditional representations of multivariate distributions require that all random variables have the *same* marginals: so if we fit a multivariate normal across a set of random variables, we are forced to fit univariate normalise to each of the marginals. It is obvious that this straightjacket is often extremely unsatisfactory (e.g., when aggregating across different risks), and copulas enable us to escape from it.
- Copulas also provide greater flexibility in that they allow us a much wider range of possible dependence structures. Imagine we have a set of marginals of a given type (e.g., normal). The traditional representation only allows us one possible type of dependence structure, a multivariate version of the corresponding univariate distribution (e.g., a multivariate normal, if our marginals are normal). However, copulas still allow us the same dependence structure if we wish to apply it (i.e., through a Gaussian copula), but also allow us a great range of additional dependence structures (e.g., through Archimedean copulas).

These advantages (and others besides) imply that copulas provide a superior approach to the modelling of multivariate statistical problems[10].

3 VaR and Monte Carlo simulating analysis

[9] As one can easily deduce from equation (2.1.2).

[10] http://www.fenews.com/embrechts

Due to its simplicity, but also because of regulatory reasons, Value-at Risk (VaR) remains one of the most popular risk measures. The emergence of Value-at-Risk (VaR) is dated away from the early nineties and its various generalisations and refinements more recently. Value-at-Risk (VaR) calculations are usually based on a Monte Carlo simulation. In this paper, we show how may be used copula methodology for the Monte Carlo Analysis.

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The basis concept behind the Monte Carlo approach is to simulate repeatedly a random process for the financial variable of interest, covering a wide range of possible situations. These variables are drawn from specified probability distributions that are assumed known. Thus simulations recreate the entire distribution of portfolio values and a distribution of portfolio values is obtained. Ordering the changes in portfolio value from worst to best, the 99% VaR, for example, is computed as the loss such that 1% of the profits or losses are bellow it, and 99% are above it.

The key to a meaningful implementation of Monte Carlo simulation is making reliable judgements about which statistical distribution is appropriate for which risk factors and estimating the parameters of the selected distributions. In practice, a wide array of distributions can be used for different risk factors. Some of the commonly used distributions are the normal, the lognormal, GARCH, and so on. An important issue is the specification of a modelling structure that meaningfully takes into account the interrelationships between different risk factors, for example, interest rates, exchange rates and so on. This multivariate simulation process captures and maintains the dependence structure of the risk factors modelled separately. To accomplish this, the simulation engine uses a framework based on the statistical concept of a copula. A copula is a function that combines marginal distributions of the variables (risk factors) into a specific multivariate distribution in which all of its one-dimensional marginals are the cumulative distribution functions (CDFs) of the risk factors (see section 2 of this paper).

Be aware that a Monte Carlo Analysis consisting of two independent simulations based only on the individual marginal distributions of the risk factors will ignore the correlation between the two risk factors and potentially provides misleading simulation results. In extreme cases, a Monte Carlo Analysis that ignores the correlation between risk factors can simulate impossible market states.

In this paper we describe how the copula methodology may be used in Monte Carlo Analysis. Our Monte Carlo Analysis contains for example risk factors interest rate and exchange rates, and their respective simulation models.

Models fitted for use in Monte Carlo simulation can be univariate equations or multivariate systems. For example, an exchange rate model might be univariate, but yield curve models are often multivariate. A typical equations are:

$$\boldsymbol{y_i} = \boldsymbol{f_i}(\boldsymbol{x}, \boldsymbol{y}, \theta_{\boldsymbol{i}}) + \varepsilon_{\boldsymbol{i}}, \tag{3.1}$$

where $i = 1,2, \ldots, N_{en}$. N_{en} is the number of endogenous variables, $\boldsymbol{y}$ is a vector of the endogenous variables, $\boldsymbol{x}$ is a vector of the exogenous variables, θ are the estimated parameters, and ε are the residual errors. The residual have a user-specified distribution function $\boldsymbol{F_i}(.)$ ($\varepsilon_{\boldsymbol{i}} \sim \boldsymbol{F_i}(\chi_{\boldsymbol{i}})$). Alternatively, equations can also be written in the general form:

$$\boldsymbol{g}_i(\boldsymbol{y}, \boldsymbol{x}, \theta) = \varepsilon_i \tag{3.2}$$

The modelling subsystem makes it possible for the user to estimate from the data the parameter vectors $\theta_{\boldsymbol{i}}$ and $\chi_{\boldsymbol{i}}$ for the equations (3.2) in each model.

Modelling a system of variables accurately is a difficult task. The underlying, ideal, distributional assumptions for each variable are usually different from each other. An individual variable may be best modelled as a *t*-distribution or as a Poisson process. The correlation of the various variables is very important to estimate as well. A joint estimation of a set of variables would make it possible to estimate a correlation structure but would restrict the modelling to single, simple multivariate distribution (for example, the normal). Even with a simple multivariate distribution, the joint estimation would be computationally difficult and would have to deal with issues of missing data.

They are known a variety of estimation methods including the Generalized Method of Moments (GMM) and Maximum Likelihood Estimation (MLE) for selecting a appropriate models to fitting risk factors. Various diagnostics and goodness of fit measures are also known, to enable the user to iteratively refine the model

specification. When an appropriately fitted model is available, that model can be later use in Monte Carlo simulation.

SAS® Risk Dimensions has a unique capability to meaningfully integrate into a single joint distribution, the marginals distributions that have been estimated in separate models, many of which might have very different distributional assumptions. The statistical construct of a copula is used for this purpose.

After the fitted models that will be used are identified, the measures of dependence that define the chosen copula $C(.)$ are estimated at the beginning of the simulation process. The estimate of the joint distribution $H(.)$ for the error vector can then be constructed as: $\boldsymbol{F}(\varepsilon) = \boldsymbol{C}(\boldsymbol{F}_1^{-1}(\varepsilon_1),\ldots,\boldsymbol{F}_n^{-1}(\varepsilon_n))$.

This fixes the distribution for ***y***, the endogenous variables, because the parameters of the model equations have already been estimated, and the equations can be used to solve for ***y***. Note that this solution process is a joint equation solution that naturally accounts for interdependencies, such as the appearance of some ***y***'s (endogenous variables) in the model equations for other ***y***'s.

Any given joint distribution function has an associated copula that is induced by it by construction. In the case of the copula induced by the multivariate normal, the simulation scheme reduces to the following steps:

1. A correlation matrix **R** is estimated from the estimated model residuals that have been made "normal" using the C.D.F.s, $\boldsymbol{F}_i$ (.), along with the inverse standard normal C.D.F., $\Phi^{-1}(.)$, which uses the relationship $\Phi^{-1}(\boldsymbol{F}_i(\varepsilon_{it}))$.
2. Using a random-number generating method that is chosen by the user from the available alternatives, independent $N(0,1)$ variables are generated and transformed to a correlated set by using **R**. They are then transformed back to the uniform by using $\Phi^{-1}(.)$. Now we have a set of non-independent uniforms, u_i, that are dependent through the copula.
3. These u_i can be transformed into a set of draws from the joint distribution by using $\varepsilon_i = \boldsymbol{F}_i^{-1}(\boldsymbol{u}_i)$. Using the model equations, the endogenous variable values for that time step for that simulation draw can then be computed.

When the copula is induced by the normal distribution, the key measure of dependence is the correlation matrix **R**, which is described above. Risk Dimensions is capable of factoring in the uncertainty in the estimation of the correlation matrix, by using a Wishart distribution that is centered around the estimated **R** to randomize the actual correlation matrix that is used in each simulation draw.

Monte Carlo simulations are based on random draws ε from a variable with the desired probability distribution. The numerical analysis usually proceeds in two steps.

The first building block for a random number generator is a uniform distribution over the interval [0,1], which produces a random variable *X*.

The next step is to transform the uniform random number *X* into the desired distribution through the inverse cumulative probability distribution function (PDF). Take the normal distribution.

4 Non commutative joint distributions

The base model for theory of copulas is in the classical probability space (Ω, S, P), where (Ω, S) is a measurable space and P is a probability measure (σ-additive function from $S \to [0,1]$)Let ξ_1, ξ_2 be some random variables on (Ω, S, P). Then for the joint distribution $F_{12}(t_1,t_2) = P(\xi_1 < t_1, \xi_2 < t_2)$ is valid $F_{12}(t_1,t_2) = P(\xi_1 < t_1, \xi_2 < t_2) = F_{21}(t_2,t_1)$, for each $t_1, t_2 \in R$. It is well known fact, that in this case we cannot study for example causality models. It means, that it is impossible describe the following situation: result of the random variable ξ_1 dependence on the random variable ξ_2 but the random variable ξ_2 does not dependent on the variable ξ_1. It is the same as $F_{12}(t_1,t_2) \neq F_{21}(t_2,t_1)$. Such situation is possible to describe by using quantum probability models.

In 1933 von Neumann has suggested the other model (von Neumann algebra). In these days they are studied several algebraic structures, which can be use for describing non-compatible random event. This problem has

been studied by several authors [1]-[8],[10]-[23]. For example, on an orthomodular lattice (an OML) L we can study causality. An OML has the following property

$$a = (a \wedge b) \vee (a \wedge b^{\perp}) \text{ iff } L \text{ is Bolean algebra.}$$

Indeed $a \geq (a \wedge b) \vee (a \wedge b^{\perp})$ for $a, b \in L$.

In the following part of this paper we try to explain, how is possible to define a joint distribution on an OML. And we introduce an example of such function, too.

Definition 4.1. [19] Let L be a nonempty set endowed with a partial ordering $\leq$. Let there exists the greatest element 1 and the smallest element 0. Let there be the lattice binary operations $\vee$, $\wedge$ and the unary operation $\perp: L \to L$ defined as follows.

i. If $a, b \in L$, then $a \vee b, a \wedge b \in L$.

ii. For any $a \in L$ $(a^{\perp})^{\perp} = a$ a $a \vee a^{\perp} = 1$.

iii. If $a, b \in L$ such that $a \leq b$, then $b^{\perp} \leq a^{\perp}$ and $b = a \vee (a^{\perp} \wedge b)$

Then $(L, 0, 1, \vee, \wedge, \perp)$ is said to be *an orthomodular lattice* (briefly *an OML*).

Let L an OML. Elements $a, b \in L$ will be called *orthogonal* ($a \perp b$) if $a \leq b^{\perp}$ and they will be called *compatible* ($a \leftrightarrow b$) if $a = (a \wedge b) \vee (a \wedge b^{\perp})$

Definition 4.2. A map $m : L \times L : [0,1]$, such that $m(0) = 0$, $m(1) = 1$ and $m(a \vee b) = m(a) + m(b)$ if $a \perp b$, is called a state on L.}

Definition 4.3. [15] Let L be an OML. The map $p : L \times L \to [0,1]$ will be called *an s-map* if the following conditions hold:

(s1) $p(1,1) = 1$;

(s2) if $a \perp b$, then $p(a,b) = 0$;

(s3) if $a \perp b$, then for any $c \in L$

$$p(a \vee b, c) = p(a,c) + p(b,c)$$
$$p(c, a \vee b) = p(c,a) + p(c,b)$$

We say that a s-map p is non-commuting, if there exist $a, b \in L$, such that $p(a,b) \neq p(b,a)$. If for each $a, b \in L$ $p(a,b) = p(b,a)$, then we say that p is commuting. We can see that the s-map has the similarly properties as a probability of intersection for two random events in the classical probability space. It has only one different property: an s-map can be non-commuting. But for compatible elements, we have the same situation, as in the classical theory. This has been studied in the following papers [12]-[17]. In this case it is possible to find for a non-commuting s-map such elements $a, b \in L$, for that

$$p(a,b) = p(a,a).p(b,b)$$
$$p(b,a) \neq p(a,a).p(b,b).$$

We will write $a \approx_p b$ and $b \not\approx_p a$. Moreover a map $\nu(a) = p(a,a)$ is a state on L.

Let $B(R)$ be a σ-algebra of Borel sets. A homomorphism $x : B(R) \to L$ is called *an observable* on L (analogical notion to a random variable). If x is an observable, then $R(x) = \{x(E); E \in B(R)\}$ is called *a range* of the observable x. It is clear that $R(x)$ is a Boolean algebra [23]. *A spectrum* of an observable x is defined by the following way: $\sigma(x) = \bigcap \{E \in B(R); x(E) = 1\}$.

We say that x and y are *compatible* ($x \leftrightarrow y$) if there exists a Boolean sub-algebra $B \subset L$ such that $R(x) \bigcup R(y) \subset B$. In other words $x \leftrightarrow y$ if for any $E,.F \in B(R)$, $x(E) \leftrightarrow y(F)$

We call an observable x a finite if $\sigma(x)$ is a finite set. Let us denote $\Im$ the set of all finite observables on L.

Let $x, y \in \Im$. Then a map $p_{x,y} : B(R) \times B(R) \to [0,1]$, such that

$$p_{x,y}(E,F) = p(x(E), y(F)),$$

is called *a joint distribution for the observables* x, y. Let us denote

$$p(x,y) = \sum_{x_i \in \sigma(x)} \sum_{y_j \in \sigma(y)} x_i y_j \, p(x(x_i), y(y_j)).$$

Then

$$p(x,x) = \sum_{x_i \in \sigma(x)} x_i^2 \, p(x(x_i), x(x_i)) = \sum_{x_i \in \sigma(x)} x_i^2 \nu(x(x_i))$$

From analogy with the classical theory of probability we can define notions for example as covariance ($c(.,.)$) and correlation coefficient ($r(.,.)$ by the following way:

$$c(x,y) = p(x,y) - \nu(x)\nu(y)$$

$$r(x,y) = \frac{c(x,y)}{\sqrt{c(x,x)c(y,y)}}$$

In spite of the classical theory of probability in this case $c(x,y)$ is not equal to $c(y,x)$ in general.

Proposition 4.2. [16] Let L be an OML, let p be an s-map on L. For each $x, y \in \Im$ there exist probability spaces (Ω_i, S_i, P_i), ($i = 1,2$), and random variables η_i, ξ_i, which are S_i-measurable such that:

a) $Ex(\eta_i) = \nu(x)$ and $Ex(\xi_i) = \nu(y)$, ($i = 1,2$), where $Ex(\varsigma)$ is expectation of random variable ς;

b) $c(x,y) = \mathrm{cov}(\eta_1, \xi_1)$ and $c(y,x) = \mathrm{cov}(\xi_2, \eta_2)$;

c) $c(x,y) \le \sqrt{c(x,x)c(y,y)}$;

d) $r(x,y) \in [-1,1]$;

if $x \leftrightarrow y$, then $c(x,y) = c(y,x)$.

From the previous it follows that a non-commuting covariance is dependent only on a non-commuting s-map. Let (Ω, S, P) be a classical probability space and η, ξ be some random variables on it. From the classical theory of probability we know, that the set of all random variables is a linear space, the covariance $\mathrm{cov}(\eta, \xi)$ is the inner product and the standard deviation. From it follows that the correlation coefficient

$$\rho(\eta, \xi) = \cos(\beta_P)$$

where β_P is „the angle“ between the random variables η, ξ in this geometry.

Example 4.3. Let $L = \{0,1, a, a^\perp, b, b^\perp c, c^\perp\}$. In this case L is the prime sum of three Boolean algebras, where a, b, c are atoms. In the following table we can see s-map $p(u,v)$, where u is in rows and v is in columns.

$p(u,v)$	a	$a^\perp$	b	$b^\perp$	c	$c^\perp$
a	0.2	0	0.06	0.14	0.08	0.12
$a^\perp$	0	0.8	0.24	0.56	0.32	0.48

b	0.1	0.2	0.3	0	0.21	1.28
$b^{\perp}$	0.1	0.6	0	0.7	0.18	0.42
C	0.1	0.3	0.2	0.2	0.4	0
$c^{\perp}$	0.1	0.5	0.1	0.5	0	0.6

It is easy to see, that $p(u,v)$ is s-map. Moreover

$$p(a,b) = p(a,a)p(b,b), \quad p(a,c) = p(a,a)p(c,c), \quad p(b,c) = p(b,b)p(c,c)$$

and

$$p(b,a) \neq p(b,b)p(a,a), \quad p(c,a) \neq p(c,c)p(a,a), \quad p(c,b) \neq p(c,c)p(b,b).$$

Let us denote $B_u = \{0,1,u,u^{\perp}\}$. B_u is a Boolean sub-algebra of L. This system L of Boolean algebras B_u ($u \neq 0,1$) is causal system endowed with the s-map p or with the conditional state f. We can imagine influence among these Boolean algebras as follows

$$B_a \approx_p B_b, \quad B_a \approx_p B_c, \quad B_b \approx_p B_c$$
$$B_b \not\approx_p B_a, \quad B_c \not\approx_p B_a, \quad B_c \not\approx_p B_b.$$

On the other hand, if we put $p_1(b,a) = p_1(a,b) = p(a,b)$,…, then p_1 is the commuting s-map. And so, there is not causality between Boolean algebras.

Conclusion.

If we compare the last part with the previous, we could see, that it is possible to study together also causality data. On the other hand it could be a danger to use some statistical methods without knowledge about role of the time in using database. It will be interesting to compare for example Granger causality with this approach.

References:

[1] Beltrametti E., Remarks on conditioning, Preprint, November, (2005).
[2] Cassinelli G., Zanghi N., *Conditional probabilities in quantum mechanics II., Il* Nuovo Cimento, vol 798, 141 – 154, (1984)
[3] Cassinelli G., Truini P., *Conditional probabilities on orthomodular lattices*, Rep. Math. Phys., vol 20, 41 – 52, (1984)
[4] Dvurečenskij A., Pulmannová S., *New Trends in Quantum Structures*, Kluwer Acad. Publ., (2000).
[5] Gudder S.,P., *Quantum probability spaces*, AMS, vol 21, 196-302, (1996)
[6] Khrennikov, A. Yu., Smoljanov, O., G., Truman, A., Kolmogorov Probability Spaces Describing Accardi Models of Quantum Correlationds , Open Sys. & . Information Dyn., Springer, 12, 371-384, (2005)
[7] Khrenikov A. Yu, *Origin of quantum probabilities*. Proc. Conf. "Foundations of Probability and Physics", Vaxjo, Sweden; p. 180-200, (2000)
[8] Khrennikov A. Yu., *Interpretations of Probability*, VSP Int. Sc. Publ., Utrecht, (1999).
[9] Kolmogoroff, A. N. Grundbegriffe der Wahrscheikchkeitsre\-chnung, Springer, Berlin, (1933).
[10] Kôpka F., Chovanec F., *D-poset*, Math. Slovaca, vol. 44, 789-807, (1984)
[11] Chovanec F., Rybáriková E., *Ideal* and *Filters in D-poset*, Int. Journ. Of Thoer. Phys. vol. 37, 17-22, (1988)
[12] Nánásiová O., *On conditional probabilities on quantum logic*., Int. Jour. of Theor. Phys., vol 25, 155 – 162, (1987)
[13] Nánásiová O., *Principle conditioning*, Int. Jour. of Theor. Phys.. vol. 43, 1383-1395, (2004)
[14] Nánásiová O. , Khrennikov A. Yu., *Observables on a quantum logic,* Foundation of Probability and Physics-2, Ser. Math. Modelling in Phys., Engin., and Cogn. Sc., vol. 5, Vaxjo Univ. Press, 417-430, (2002)
[15] Nánásiová O., Map for Simultaneus Measurements for a Quantum Logic, Int. Journ. of Theor. Phys., Vol. 42, No. 8, 1889-1903, (2003)

[16] Khrenikov A., Nánásiová O., *Representation theorem of observables on a quantum system*, Int. Jour. Theor. Phys.(2006)
[17] Nánásiová O., *Representation of conditional probability on quantum logic*, Soft Comp, (2000).
[18] Navara, M., *When a residuum can be computed from a t-norm and a negation*., Proceedings of the 3rd International Workshop on Nonstandard Logics, CTU, Praha, May 12-14, (2004)
[19] Pták P., Pulmannová S., *Quantum Logics*, Kluwer Acad. Press, Bratislava (1991).
[20] Pták P., *On prehilbert space logics,* Proceedings of the 3rd International Workshop on Nonstandard Logics, CTU, Praha, May 26-31, (2004)
[21] Riečan B., Neubrun T., *Measure theory*, Kluwer Acad. Press, Bratislava, (1997).

E-Leader, Slovakia 2006

[22] Riečanová Z., *Pták Sum of Boolean Algebra with an Effect Algebra and its Complements,* Int. Jour. of Theor. Phys. vol. 34, 1719-1727, (1995)
[23] Varadarajan V., *Geometry of quantum theory*, Princeton, New Jersey, D. Van Nostrand, (1968).
[24] Amendment to the capital accord to incorporate market risks, Basle Committee on Banking Supervision, January, (2005)
[25] A new capital adequacy framework, Basle Committee on Banking Supervision, June (1999), N° 50.
[26] Jorion, P., *Value at Risk: The Benchmark for controlling Market risk.* Blacklick, OH, USA: McGraw-Hill Professional Book Group, (2000).
[27] http://www.sas.com online documentation.
[28] Embrechts, P., H¨oing, A., Juri, A, *Using Copulae to bound the Value-at-Risk for functions of dependent risks.* http://www.gloriamundi.org.
[29] Di Clemente, A., Romano, C., *Measuring Portfolio Value-at-Risk by a Copula-EVT based approach*. http://www.gloriamundi.org, February (2003)
[30] Sklar, A., *Fonctions de répartition à n dimensions et leurs marges*, Publications de l'Institut de Statistique de l'Université de Paris, 8, 229-231, (1959)
[31] Sklar, A., *Random variables, distribution functions, and copulas – a personal look backward and forward, in Distributions with Fixed Marginals and Related Topics,* ed. By L.Rüschendorff, B. Schweizer and M. Taylor, pp. 1-14. Institute of Mathematical Statistics, Hayward, CA. (1996).
[32] Bouyé, E., Durrleman, V., Nikeghbali, A., Riboulet, G., Roncalli, T., *Copulas for Finance. A reading Guide and Some Applications.*, Groupe de Recherche Opérationelle, Crédit Lyonais , Working paper, http://www.gloriamundi.org, March (2001)
[33] Bouyé, E., Durrleman, V., Nikeghbali, A., Riboulet, G., Roncalli, T., *Copulas: an open field for risk management.*, Groupe de Recherche Opérationelle, Crédit Lyonais, Working paper, http://www.gloriamundi.org, (2000)
[34] Embrechts,P., Klüppelberg, C., Mikosch, T., *Modelling Extremal Events for Insurance and Finance,* Springer-Verlag, Berlin, (1997).
[35] Embrechts, P., McNeil, A.J., Straumann, D., *Correlation and dependency in risk management: properties and pitfalls.,* Department of Mathematik, ETHZ, Zürich, Working Paper. (2000)
[36] Li, D.X., *On default correlation: a copula function approach.* RiskMertics Group, Working Paper, first draft, (1999).
[37] Li, D.X., *On default correlation: a copula function approach.* RiskMertics Group, Working Paper, (2000).
[38] Nelsen, R.B., *An introduction to Copulas*, Lecture Notes in Statistics. Springer Verlag, New York, (1999).
[39] Embrechts, P, http://www.fenews.com