A note to Copula Functions

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Introduction

The regulatory requirements (BASEL II) cause the necessity to build sound internal models for credit, market and operational risks.

It is inevitable to solve an important problem:

"How to model a joint distribution of different risk?"

Problem

Consider a portfolio of n risks: $X_1, \ldots, X_n$. Suppose, that we want to examine the distribution of some function $f(X_1, \ldots, X_n)$ representing the risk of the future value of a contract written on the portfolio.

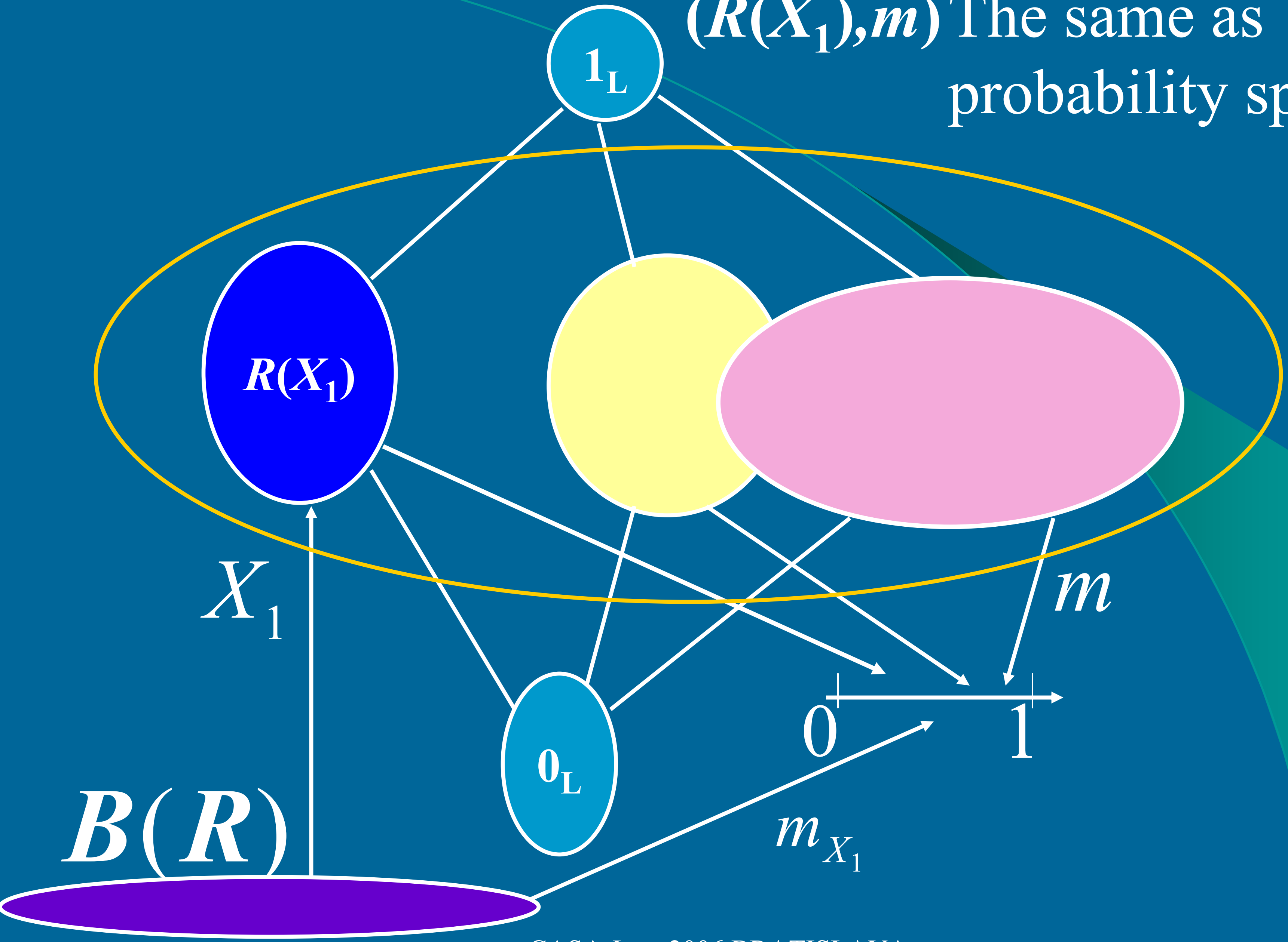
Approaches

A. Correlation
B. Copulas
C. s-maps

$(\Omega, \boldsymbol{S}, \boldsymbol{P})$ → A. Correlation, B. Copulas

$\bigcup_i (\Omega_i, \boldsymbol{S}_i, \boldsymbol{P}_i)$ → C. s-maps

$(R(X_1),m)$ The same as probability space



A. Correlation

- Estimate marginal distributions $F_1, \ldots, F_n$. (They completely determines the dependence structure of risk factors)
- Estimate pair wise linear correlations $\rho(X_i, X_j)$ for $i,j \in \{1,\ldots,n\}$ with $i \neq j$
- Use this information in some Monte Carlo simulation procedure to generate dependent data

Common approach

Common methodologies for measuring portfolio risk use the multivariate conditional Gaussian distribution to simulate risk factor returns due to its easy implementation.

Empirical evidence underlines its inadequacy in fitting real data.

B. Copula approach

- Determine the margins $F_1, \ldots, F_n$, representing the distribution of each risk factor, estimate their parameters fitting the available data by soundness statistical methods (e.g. GMM, MLE)
- Determine the dependence structure of the random variables $X_1, \ldots, X_n$, specifying a meaningful copula function

Copula ideas provide

- a better understanding of dependence,
- a basis for flexible techniques for simula-ting dependent random vectors,
- scale-invariant measures of association similar to but less problematic than linear correlation,

Copula ideas provide

- a basis for constructing multivariate distributions fitting the observed data
- a way to study the effect of different dependence structures for functions of dependent random variables, e.g. upper and lower bounds.

Definition 1: An n-dimensional copula is a multivariate C.D.F. $\boldsymbol{C}$, with uniformly distributed margins on [0,1] ($\boldsymbol{U}$(0,1)) and it has the following properties:

1. $\boldsymbol{C}: [0,1]^n \to [0,1]$;
2. $\boldsymbol{C}$ is grounded and n-increasing;
3. $\boldsymbol{C}$ has margins $\boldsymbol{C_i}$ which satisfy

$$\boldsymbol{C_i}(u) = \boldsymbol{C}(1, \ldots, 1, u, 1, \ldots, 1) = u$$

for all $u \in [0,1]$.

Sklar's Theorem

Theorem: Let F be an n-dimensional C.D.F. with continuous margins $F_1, \ldots, F_n$. Then F has the following unique copula representation:

$$F(x_1,\ldots,x_n)=C(F_1(x_1),\ldots,F_n(x_n)) \quad (2.1.1)$$

Corollary: Let F be an n-dimensional C.D.F. with continuous margins $F_1, \ldots, F_n$ and copula **C** (satisfying (2.1.1)).

Then, for any $\boldsymbol{u}=(u_1, \ldots, u_n)$ in $[0,1]^n$:

$$\boldsymbol{C(u_1,\ldots,u_n) = F(F_1^{-1}(u_1),\ldots,F_n^{-1}(u_n))} \tag{2.1.2}$$

Where $\boldsymbol{F_i^{-1}}$ is the generalized inverse of $\boldsymbol{F_i}$.

Corollary: The Gaussian copula is the copula of the multivariate normal distribution. In fact, the random vector $\boldsymbol{X}=(X_1,\ldots,X_n)$ is multivariate normal iff:

1) the univariate margins $\boldsymbol{F}_1, \ldots, \boldsymbol{F}_n$ are Gaussians;

2) the dependence structure among the margins is described by a unique copula function $\boldsymbol{C}$ (the normal copula) such that:

$$\boldsymbol{C}_R^{Ga}(u_1,\ldots,u_n)=\Phi_R(\phi_1^{-1}(u_1),\ldots,\phi_n^{-1}(u_n)), \quad (2.1.3)$$

where Φ_R is the standard multivariate normal C.D.F. with linear correlation matrix **R** and ϕ^{-1} is the inverse of the standard univariate Gaussian C.D.F.

1. Traditional versus Copula representation

- Traditional representations of multivariate distributions require that all random variab-les have the *same* marginals
- Copula representations of multivariate dis-tributions allow us to fit any marginals we like to different random variables, and these distributions might differ from one variable to another

2. Traditional versus Copula representation

- The traditional representation allows us only one possible type of dependence structure
- Copula representation provides greater flexibility in that it allows us a much wider range of possible dependence structures.

Software

- Risk Metrics system uses the traditional approache
- SAS Risk Dimension software use the Copula approache

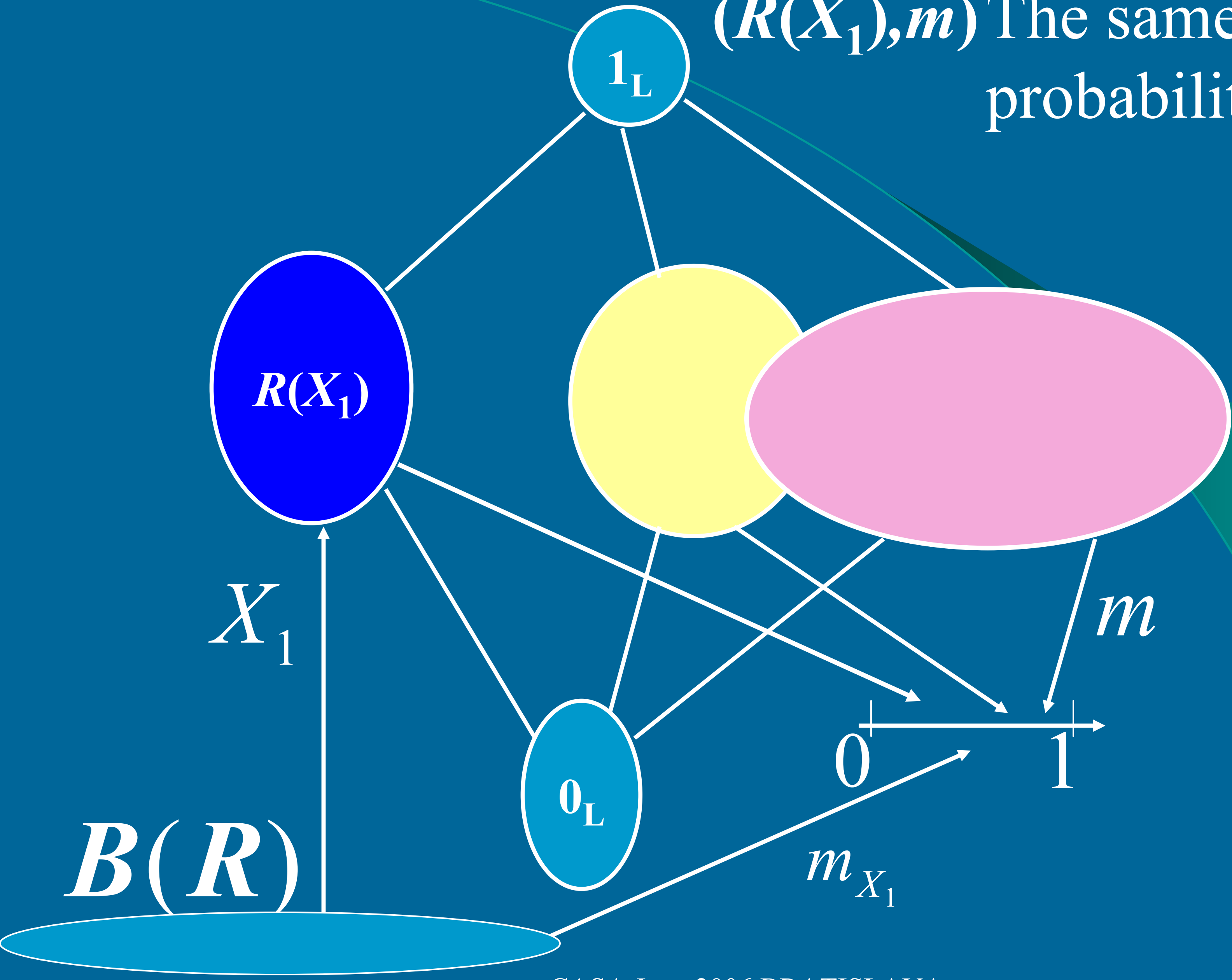
C. s-map

- An orthomodular lattice (OML) $(\boldsymbol{L}, \leq, 1_L, 0_L, \vee, \wedge, \perp)$
- $\boldsymbol{b}, \boldsymbol{a} \in \boldsymbol{L}$ are called orthogonal $\boldsymbol{a} \leq \boldsymbol{b}^{\perp}$
- $b, a \in L$ are called compatible $\boldsymbol{a} \geq (\boldsymbol{a} \wedge \boldsymbol{b}) \vee (\boldsymbol{a} \wedge \boldsymbol{b}^{\perp})$
- a state $m{:}L \longrightarrow [0,1]$

1. $m(1_L) = 1;$
2. *m is additive*

Boolean algebra $\longleftrightarrow$ $\boldsymbol{a} = (\boldsymbol{a} \wedge \boldsymbol{b}) \vee (\boldsymbol{a} \wedge \boldsymbol{b}^{\perp})$

$(R(X_1),m)$ The same as probability space



S-map and conditional state on an OML

- S-map: map from

  $p: L^n \rightarrow [0,1]$

1. additive in each coordinate;
2. if there exist orthogonal elements, then = 0;

- Conditional state

  $f: LxL_0 \rightarrow [0,1]$

1. additive in the first coordinate;
2. Theorem of full probability

Non-commutative s-map

$$p(a,b) \neq p(b,a)$$

$$a = (a \wedge b) \vee (a \wedge b^{\perp})$$

$$p(a,b) = p(b,a)$$

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Thank you for your kindly attention